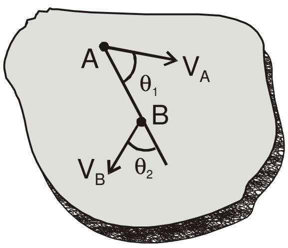
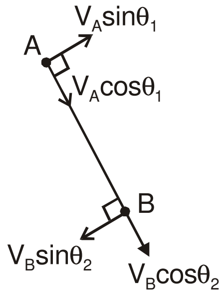
RIGID BODY DYNAMICS

1. RIGID BODY :

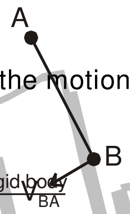
Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time) that means the shape and size do not change, during the motion. Eg : Fan, Pen, Table, stone and so on.
Our body is not a rigid body, two blocks with a spring attached between them is also not a rigid body.
For every pair of particles in a rigid body, there is no velocity of seperation or approach between the particles. In the figure shown velocities of A and B with respect ground are V_A and V_B respectively.





If the above body is rigid

$$V_A \cos\theta_1 = V_B \cos\theta_2$$



NOTE : With respect to any particle of rigid body the motion of any other particle of that rigid body is circular.
V_{BA} = relative velocity of B with respect to A.

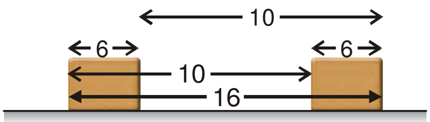
Types of Motion of rigid body

- Pure Translational Motion
- Pure Rotational Motion
- Combined Translational and Rotational Motion

I. Pure Translational Motion :

A body is said to be in pure translational motion if the displacement of each particle is same during any time interval howsoever small or large. In this motion all the particles have same $\vec{s}, \vec{v}$ & , $\vec{a}$ at an instant.

Ex : A box is being pushed on a horizontal surface.

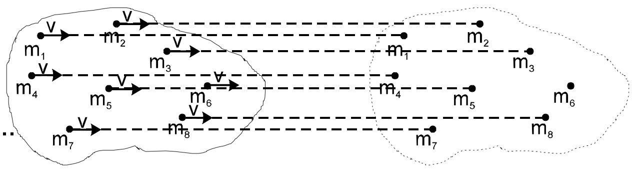


$\vec{V}_{cm} = \vec{V}$ of any particle

$\vec{a}_{cm} = \vec{a}$ of any particle

$\Delta\vec{S}_{cm} = \Delta\vec{S}$ of any particle

For pure translation motion :–



$$\vec{F}_{ext} = + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Where $m_1, m_2, m_3, \ldots\ldots$ are the masses of different particles of the body having accelerations $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots\ldots\ldots$respectively

But acceleration of all the particles are same so $\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \ldots\ldots = \vec{a}$ $= M\vec{a}$

Where M = Total mass of the body

$\vec{a}$ = acceleration of any particle or of centre of mass or of body

$= m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ...$

Where m_1, m_2, m_3,....... are the masses of different particles of the body having velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3$ ....respectively

But velocities of all the particles are same so $\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = ... = \vec{v}$

$= M\vec{v}$

Where $\vec{v}$ = velocity of any particle or of centre of mass.

Total Kinetic Energy of body $= \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2 + ... = \frac{1}{2} Mv^2$

II. Pure Rotational Motion :

A body is said to be in pure rotational motion if the perpendicular distance of each particle remains constant from a fixed line or point and do not move parallel to the line, and that line is known as axis of rotation. In this motion all the particles have same $\vec{\theta}, \vec{\omega}$ & , $\vec{\alpha}$ at an instant. Eg :– a rotating ceiling fan, arms of a clock.

For pure rotation motion : –

$\theta = \frac{s}{r}$ Where θ = angle rotated by the particle

s = length of arc traced by the particle.

r = distance of particle from the axis of rotation.

$\omega = \frac{d\theta}{dt}$ Where ω = angular speed of the body.

$\alpha = \frac{d\omega}{dt}$ Where α = angular acceleration of the body.

All the parameters θ, ω and α are same for all the particles.

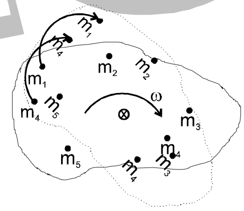
Axis of rotation is perpendicular tothe plane of rotationof particles.

Special case : If α = constant ,

$\omega = \omega_0 + \alpha t$ Where ω_0 = initial angular speed

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ t = time interval

$\omega^2 = \omega_0^2 + 2\alpha\theta$



Total Kinetic Energy $= \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2 + ...$

$= \frac{1}{2} [m_1r_1^2 + m_2r_2^2 + ...] \omega^2$

$= \frac{1}{2} I\omega^2$ Where I = Moment of Inertia $= m_1r_1^2 + m_2r_2^2 + ...$

ω = angular speed of body.

Ex.1 A bucket is being lowered down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that the rope does not slip on the pulley. Find the angular velocity and angular acceleration of the pulley at an instant when the bucket is going down at a speed of 20 cm/s and has an acceleration of 4.0 m/s².

Sol. Since the rope does not slip on the pulley, the linear speed v of the rim of the pulley is same as the speed of the bucket.

The angular velocity of the pulley is then

$$\omega = v/r = \frac{20\,cm/s}{10\,cm} = 2 \text{ rad/s}$$

and the angular acceleration of the pulley is

$$\alpha = a/r = \frac{4.0\ m/s^2}{10\ cm} = 40 \text{ rad/s}^2.$$

Teko Classes, Maths : Suhag R. Kariya (S. R. K. Sir), Bhopal Phone : 0 903 903 7779, 0 98930 58881.

Q.1 A disc of radius 10 cm is rotating about its axis at an angular speed of 20 rad/s. Find the linear speed of
(a) a point on the rim,
(b) the middle point of a radius

Ans. 2 m/s, 1 m/s

Ex.2 A wheel rotates with a constant acceleration of 2.0 rad/s^2. If the wheel starts from rest, how many revolutions will it make in the first 10 seconds?

Sol. The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.0\ \text{rad/s}^2)(10\ \text{s})^2 = 100\ \text{rad}.$$

As the wheel turns by 2π radian in each revolution, the number of revolutions in 10 s in

$$n = \frac{100}{2\pi} = 16.$$

Ex.3 The wheel of a motor, accelerated uniformly from rest, rotates through 2.5 radian during the first second. Find the angle rotated during the next second.

Sol. As the angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2.$$

Thus, $2.5\ \text{rad} = \frac{1}{2}\alpha(1\text{s})^2$

$\alpha = 5\ \text{rad/s}^2$ or $\alpha = 5\ \text{rad/s}^2$

The angle rotated during the first two seconds is

$$= \frac{1}{2}\times(5\ \text{rad/s}^2)(2\text{s})^2 = 10\ \text{rad}.$$

Thus, the angle rotated during the 2nd second is
10 rad – 2.5 rad = 7.5 rad.

Q.2 A wheel is making revolutions about its axis with a uniform angular acceleration. Starting from rest, it reaches 100 rev/sec in 4 seconds. Find the angular acceleration. Find the angle rotated during these four seconds.

Ans. 25 rev/s^2, 400 π rad

Ex.4 Starting from rest, a fan takes five seconds to attain the maximum speed of 400 rpm (revolution per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.

Sol. Let the angular acceleration be α. According to the question,
400 rev/min = 0 + α 5 s ....(i)
Let t be the time taken in attaining the speed of 200 rev/min which is half the maximum.
Then, 200 rev/min = 0 + αt ....(ii)
Dividing (i) by (ii), we get,
2 = 5 s/t or t = 2.5 s.

Ex.5 The motor of an engine is rotating about its axis with an angular velocity of 100 rev/minute. It comes to rest in 15 s, after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.

Sol. The initial angular velocity = 100 rev/minute = $(10\pi/3)$ rad/s.
Final angular velocity = 0.
Time invertial = 15 s.
Let the angular acceleration be α. Using the equation $\omega = \omega_0 + at$, we obtain
$\alpha = (-2p/9)\ \text{rad/s}^2$
The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= \left(\frac{10\pi}{3}\frac{\text{rad}}{\text{s}}\right)(15\text{s}) - \frac{1}{2}\left(\frac{2\pi}{9}\frac{\text{rad}}{\text{s}^2}\right)(15\text{s})^2 = 25\pi\ \text{rad} = 12.5\ \text{revolutions}.$$

Hence the motor rotates through 12.5 revolutions before coming to rest.

III. Combined Translational and Rotational Motion :

A body is said to be in translation and rotational motion if all the particles rotates about an axis of rotation and the axis of rotation moves with respect to the ground.

2. MOMENT OF INERTIA (I) :

Definition : Moment of Inertia is defined as the capability of system to oppose the change produced in the rotational motion of a body.

Moment of Inertia depends on :

(i) density of the material of body
(ii) shape & size of body
(iii) axis of rotation

Combinedly we can say that it depends upon distribution of mass relative to axis of rotation.

Note : Moment of inertia does not change if the mass :
(i) is shited parallel to the axis of the rotation.
(ii) is rotated with constant radius about axis of rotation.

Moment of Inertia is a scalar positive quantity.

$$I = mr_1^2 + m_2r_2^2 + \ldots\ldots\ldots\ldots$$
$$= I_1 + I_2 + I_3 + \ldots\ldots\ldots\ldots$$

SI units of Moment of Inertia is Kgm^2.

Moment of Inertia of :

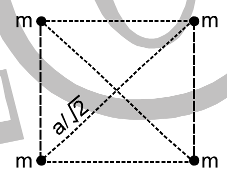
(I) A single particle : $I = mr^2$

where m = mass of the particle
r = perpendicular distance of the particle from the axis about which moment of Inertia is to be calculated

(II) For many particles (system of particles) :

$$I = \sum_{i=1}^{n} m_i r_i^2$$

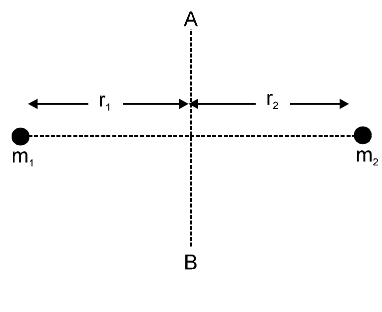
Ex.6 Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.



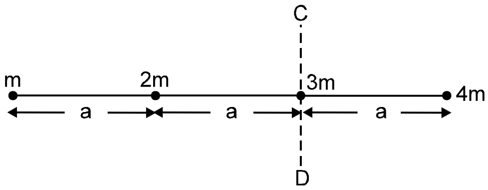
Sol. The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore, $4 \times \frac{1}{2}ma^2 = 2ma^2$.

Ex.7 Two heavy particles having masses m_1 & m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively.
(i) What is the moment of inertia of the system about axis AB ?
(ii) What is the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2 ?
(iii) What is the moment of inertia of the system about an axis passing through m_1 and m_2 ?

Sol. (i) Moment of inertia of particle on left is $I_1 = m_1r_1^2$.
Moment of Inertia of particle on right is $I_2 = m_2r_2^2$.
Moment of Inertia of the system about AB is $I = I_1 + I_2 = m_1r_2^2 + m_2r_2^2$
(ii) Moment of inertia of particle on left is $I_1 = 0$
Moment of Inertia of particle on right is $I_2 = m_2(r_1 + r_2)^2$.
Moment of Inertia of the system about AB is $I = I_1 + I_2 = 0 + m_2(r_1 + r_2)^2$
(iii) Moment of inertia of particle on left is $I_1 = 0$
Moment of Inertia of particle on right is $I_2 = 0$
Moment of Inertia of the system about AB is $I = I_1 + I_2 = 0 + 0$

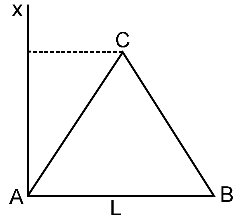


Q.3 Four point masses are connected by a massless rod as shown in figure. Find out the moment of inertia of the system about axis CD ?



Ans. $10\ ma^2$

Ex.8 Three particles, each of mass m, are situated at the vertices of an equilateral triangle ABC of side L (figure). Find the moment of inertia of the system about the line AX perpendicular to AB in the plane of ABC.

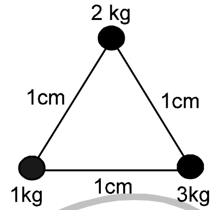


Sol. Perpendicular distance of A from AX = 0
Perpendicular distance of B from AX = L
Perpendicular distance of C from AX = L/2
Thus, the moment of inertia of the particle at A = 0, of the particle at B - mL^2, and of the particle at C = $m(L/2)^2$. The moment of inertia of the three-particle system about AX is

$$0 + mL^2 + m(L/2)^2 = \frac{5\ mL^2}{4}$$

Note that the particles on the axis do not contribute to the moment of inertia.

Q.4 Three point masses are located at the corners of an equilibrium triangle of side 1 cm. Masses are of1,2,&3kg respectively and kept as shown in figure. Calculate the moment of Inertia of system about an axis passing through 1 kg mass and perpendicular to the plane of triangle ?



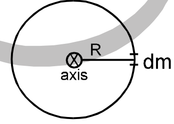
Ans. $5 \times 10^{-4}\ kgm^2$

(III) For a continuous object :

$$I = \int dmr^2$$

where dm = mass of a small element
r = perpendicular distance of the particle from the axis

Ex. 9 Calculate the moment of inertia of a ring having mass M, radius R and having uniform mass distribution about an axis passing through the centre of ring and perpendicular to the plane of ring ?



Sol. $I = \int (dm)r^2$

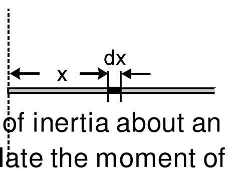
Because each element is equally distanced from the axis so r = R

$$= R^2 \int dm = MR^2$$

$I = MR^2$ (**Note :** Answer will remain same even if the mass is nonuniformly distributed because $\int dm = M$ always.)

Ex.10 Calculate the moment of inertia of a uniform rod of mass M and length ℓ about an axis passing through an end and perpendicular to the rod.

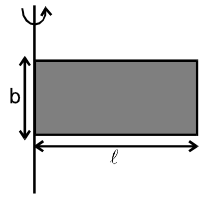
Sol. $I = \int (dm)r^2 = \int_0^{\ell} \left(\frac{M}{\ell}dx\right)x^2 = \frac{M\ell^2}{3}$



Q.5 Using the above formula or other wise find the moment of inertia about an axis which is perpendicular to the rod and passing through centre of rod. Also calculate the moment of inertia of rod about an axis passing through rod and parallel to the rod.

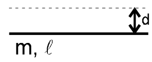
Ans. $\frac{M\ell^2}{12}$, 0

Q.6 Using the above result $\left(I = \frac{m\ell^2}{3}\right)$ or otherwise determined the moment of inertia of a uniform rectangular plate of side 'b' and 'ℓ'' about an axis passing through the edge 'b' and in the plane of plate.



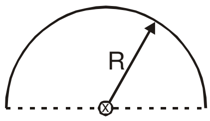
Ans. $\frac{M\ell^2}{3}$

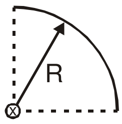
Q.7 Find the moment of inertia of uniform rod of length 'ℓ'about the axis parallel to the rod and 'd' distance apart

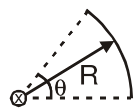


Ans. Md^2

Q.8 Find out the moment of Inertia of figures shown each having mass M, radius R and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane ?

(a) 

(b) 

(c) 

Ans. MR^2 (infact M.I. of any part of mass M of a ring of radius R about axis passing through geometrical centre and perpendicular to the plane of the ring is = MR^2)

Q.9 Find out the moment of Inertia of a hollow cyllinder of mass M, radius R and having uniform mass distribution about its axis. Will the answer change if mass is nonuniformly distributed?

Ans. MR^2, No

(IV) For a larger object : $I = \int dI_{element}$

where dI = moment of inertia of a small element

Element chosen should be :

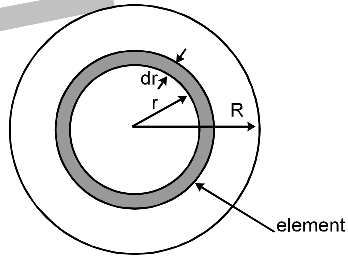
(i) as larger as possible among all types of elements.

(ii) as much symmetric as possible

(iii) moment of inertia of element should be known earlier.

Ex.11 Determine the moment of Inertia of a uniform disc having mass M, radius R about an axis passing through centre & perpendicular to the plane of disc ?

Sol. $I = \int dI_{ring}$

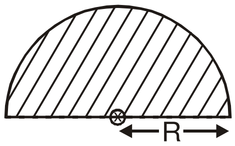


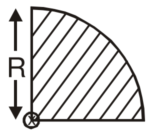
element - ring $dI = dmr^2$

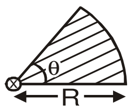
$dm = \frac{M}{\pi R^2}\, 2\pi r dr$ (here we have used the uniform mass distribution)

$\therefore \quad I = \int_0^R \frac{M}{\pi R^2}.(2\pi r dr).r^2 \quad \Rightarrow \quad I = \frac{MR^2}{2}$

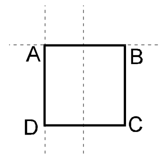
Q.10 Calculate the moment of Inertia of figure shown each having mass M, radius R and having uniform mass distribution about an axis pependicular to the plane and passing through centre?

(a) 

(b) 

(c) 

Q.11 Find the moment of inertia of the uniform square plate of side 'a' and mass M about the axis AB.



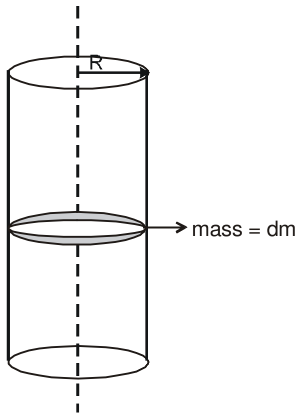
Ans. $\frac{ma^2}{3}$

Ex.12 Calculate the moment of inertia of a uniform hollow cylinder of mass M, radius R and length ℓ about its axis.

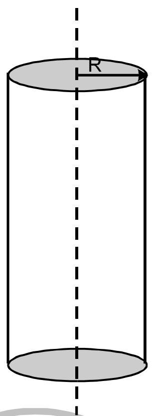
Sol. Moment of inertia of a uniform hollow cylinder is

$$I = \int (dm)R^2$$

$$= mR^2$$



Q.12 Calculate the moment of inertia of a uniform solid cylinder of mass M, radius R and length ℓ about its axis.



Ans. $I = \dfrac{MR^2}{2}$

Two Important Theorems on Moment of Inertia :

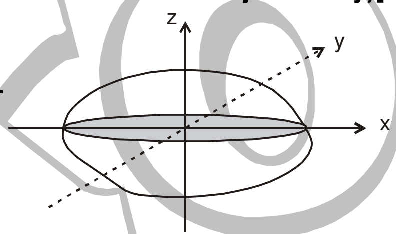
I. Perpendicular Axis Theorem [Only applicable to plane lamina (that means for 2-D objects only)].

$I_z = I_x + I_y$ (when object is in x-y plane).

Where I_z = moment of inertia of the body about z axis.
I_x = moment of inertia of the body about x axis.
I_y = moment of inertia of the body about y axis.

$I_y = I_x + I_z$ (when object is in x-z plane)

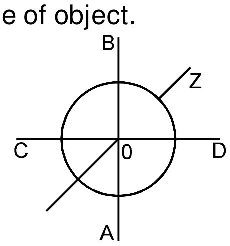
$I_x = I_y + I_z$ (when object is in y-z plane)



Body is in x-y plane

Note : Defined for any 3 perpendicular concurrent axis out of which two lie in the plane of object.

Ex.13 Find the moment of inertia of a uniform ring of mass M and radius R about a diameter.

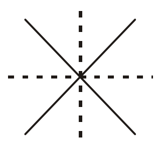


Sol. Let AB and CD be two mutually perpendicular diameters of the ring. Take them as X and Y-axes and the line perpendicular to the plane ofthe ring through the centre as the Z-axis. The moment of inertia of the ring about the Z-axis is $I = MR^2$. As the ring is uniform, all of its diameters are equivalent and so $I_x = I_y$, From perpendicular axes theorem,

$$I_z = I_x + I_y. \qquad \text{Hence} \quad I_x = \frac{I_z}{2} = \frac{MR^2}{2}.$$

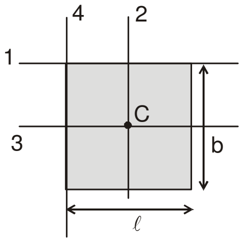
Similarly, the moment of inertia of a uniform disc about a diameter is $MR^2/4$.

Ex.14 Two uniform identical rods each of mass M and length ℓ are joined to form a cross as shown in figure. Find the moment of inertia of the cross about a bisector as shown dotted in the figure.



Sol. Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\dfrac{M\ell^2}{12}$ and hence the moment of inertia of the cross is $\dfrac{M\ell^2}{6}$. The moment of inertia of the cross about the two bisector are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\frac{M\ell^2}{12}$.

Ex.15 In the figure shown find moment of inertia of a plate having mass M, length ℓ and width b about axis 1,2,3 and 4. Assume that see is centre and mass is uniformly distributed



Sol. Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1)

$I_1 = Mb^2 / 3$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2)

$I_2 = M\ell^2 / 12$

Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3)

$I_3 = Mb^2 / 12$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4)

$I_4 = M\ell^2 / 3$

Ex. 16 Find the moment of Inertia of a uniform ring of mass M and radius R about a diameter.

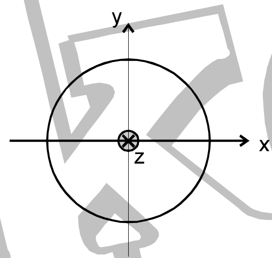
Sol. Consider x & y two mutually perpendicular diameters of the ring.

$I_x + I_y = I_z$

$I_x = I_y$ (due to symmetry)

$I_z = MR^2$

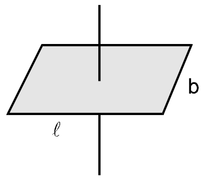
$I_x = I_y = \frac{MR^2}{2}$



Q.13 Find the moment of inertia of a uniform disc having mass M, radius R about a diameter.

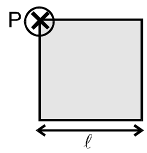
Ans. $\frac{MR^2}{4}$

Q.14 Find the moment of inertia of a uniform rectangular plate of mass M, edges of length 'ℓ' and 'b' about its axis passing through centre and perpendicular to it.



Ans. $\frac{M(\ell^2 + b^2)}{12}$

Q.15 Find the moment of inertia of a uniform square plate of mass M, edge of length 'ℓ' about its axis passing through P and perpendicular to it.



Ans. $\frac{2m\ell^2}{3}$

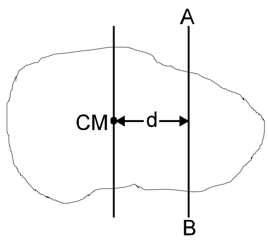
II. Parallel Axis Theorem (Applicable to any type of object):

$I_{AB} = I_{cm} + Md^2$

Where

I_{cm} = Moment of Inertia of the object about an axis passing through centre of mass and parallel to axis AB

I_{AB} = Moment of Inertia of the object about axis AB



M = Total mass of object
d = perpendicular distance between axis about which moment of Inertia is to be calculated & the one passing through the centre of mass

Ex.17 Find the moment of inertia of a solid cylinder of mass M and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.

Sol. The moment of inertia of the cylinder about its axis = $\frac{MR^2}{2}$.

Using parallel axes theorem, $I = I_0 + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$.

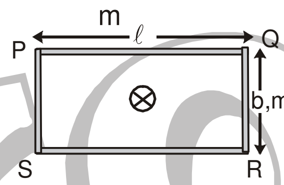
Similarly, the moment of inertia of a solid sphere about a tangent is

$$\frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

Q.16 Find out the moment of inertia of a ring having uniform mass distribution of mass M & radius R about an axis which is tangent to the ring and (i) in the plane of the ring (ii) perpendicular to the plane of the ring.

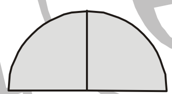
Ans. (i) $\frac{3MR^2}{2}$ (ii) $2MR^2$

Q.17 Calculate the moment of inertia of a rectangular frame formed by uniform rods having mass m each as shown in figure about an axis passing through its centre and perpendicular to the plane of frame ? Also find moment of inertia about an axis passing through PQ ?



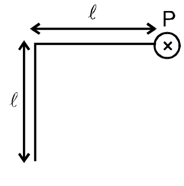
Ans : (i) $\frac{2m}{3}(\ell^2 + b^2)$ (ii) $\frac{5mb^2}{3}$

Q.18 Find out the moment of inertia of a semi circular disc about an axis passing through its centre of mass and perpendicular to the plane?



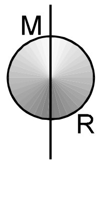
Ans. $\left[\frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2\right]$

Q.19 Find the moment of inertia of the two uniform joint rods about point P as shown in figure. Using parallel axis theorem.

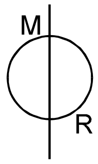


Ans. $\frac{10m\ell^2}{3}$

List of some useful formula :

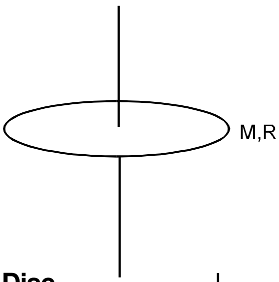
Object	Moment of Inertia
Solid Sphere 	$\frac{2}{5}MR^2$ (Uniform)

Hollow Sphere



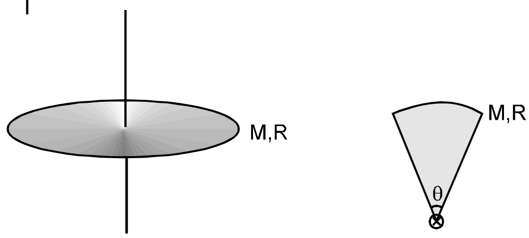
$\frac{2}{3}MR^2$ (Uniform)

Ring.



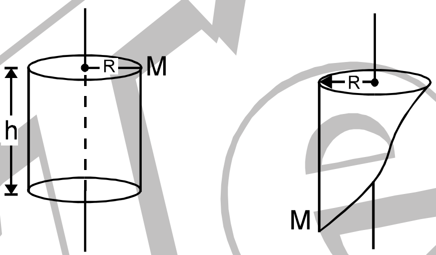
MR^2 (Uniform or Non Uniform)

Disc



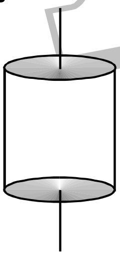
$\frac{MR^2}{2}$ (Uniform)

Hollow cylinder

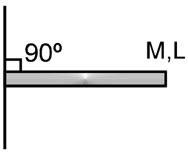


MR^2 (Uniform or Non Uniform)

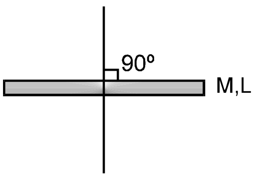
Solid cylinder



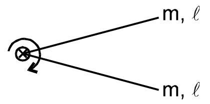
$\frac{MR^2}{2}$ (Uniform)



$\frac{ML^2}{3}$ (Uniform)

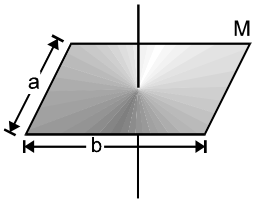


$\frac{ML^2}{12}$ (Uniform)



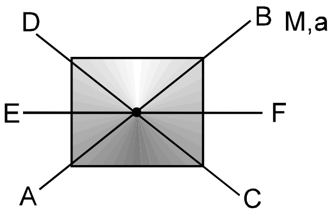
$\frac{2m\ell^2}{3}$ (Uniform)

Rectangular Plate



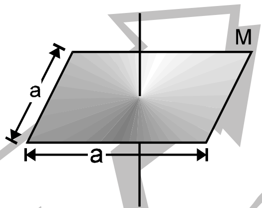
$I = \frac{M(a^2 + b^2)}{12}$ (Uniform)

Square Plate



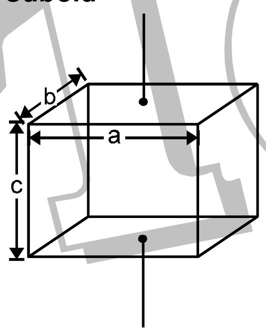
$I_{AB} = I_{CD} = I_{DF} = \frac{Ma^2}{12}$ (Uniform)

Square Plate



$\frac{Ma^2}{6}$ (Uniform)

Cuboid



$\frac{M(a^2 + b^2)}{12}$ (Uniform)

3. Radius of Gyration :

As a measure of the way in which the mass of a rotating rigid body is distributed with respect to the axis of rotation we can define a new parameter, the radius of gyration. It is related to the moment of intertia and total mass of the body.

$I = MK^2$

where I = Moment of Inertia of a body

M = Mass of a body

K = Radius of gyration

$$K = \sqrt{\frac{I}{M}}$$

Length K is the geometrical property of the body and axis of rotation.
S.I. Unit of K is meter.

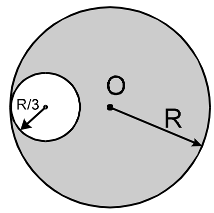
Ex.18 Find the radius of gyration of a solid uniform sphere of radius R about its tangent.

Sol. $I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2 = mK^2 \quad \Rightarrow K = \sqrt{\frac{7}{5}}R$

Q.20 Find the radius of gyration of a hollow uniform sphere of radius R about its tangent.

Ans. $\sqrt{\frac{5}{3}}R$

4. Moment of inertia of Bodies with cut :



Ex.19 A uniform disc of radius R has a round disc of radius R/3 cut as shown in Fig. .The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical centre of original disc and perpendicular to the plane of the disc.

Sol. Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$ and σ.

Now $\quad I_0 = I_\sigma + I_{-\sigma}$

$I_\sigma = (\sigma \pi R^2)R^2/2$ = M.I. of σ about o

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2]\,(2R/3)^2$$

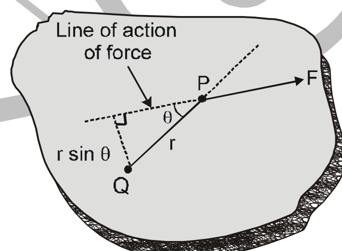
= M.I. of $-\sigma$ about o

$\therefore \quad I_0 = \frac{4}{9}\sigma\pi R^4$ **Ans.**

Q.21 Find the moment of inertia of a uniform disc of radius R_1 having an empty symmetric annular region of radius R_2 in between, about an axis passing through geometrical centre and perpendicular to the disc.

Ans. $\frac{M(R_1^{\,2} + R_2^{\,2})}{2}$

5. TORQUE :



Torque represents the capability of a force to produce change in the rotational motion of the body.

5.1 Torque about point :

Torque of force $\vec{F}$ about a point $\quad \vec{\tau} = \vec{r} \times \vec{F}$

Where $\quad \vec{F}$ = force applied

P = point of application of force

Q = Point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force from the point about which we want to determine the torque.

$|\vec{\tau}| = r F \sin\theta \qquad = r_\perp F \qquad = rF_\perp$

Where $\quad \theta$ = angle between the direction of force and the position vector of P wrt. Q.

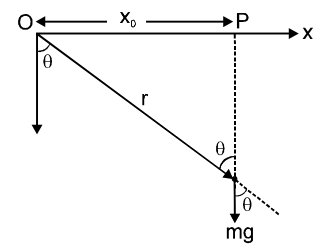
$r_\perp$ = perpendicular distance of line of action of force from point Q.

$F_\perp$ = force arm

SI unit of torque is Nm

Torque is a vector quantity and its direction is determined using right hand thumb rule.

Ex.20 A particle of mass M is released in vertical plane from a point P at $x = x_0$ on the x-axis it falls vertically along the y-axis. Find the torque τ acting on the particle at a time t about origin ?



Sol. Torque is produced by the force of gravity.

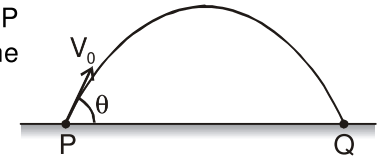
$\vec{\tau} = r F \sin\theta\ \hat{k}$

or $\quad \tau = r_\perp F = x_0 mg$

$= r\, mg\, \frac{x_0}{r} = mgx_0\, \hat{k}$

Q.22

A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the torque about the point of projection acting on the particle when
(a) it is at its maximum height ?
(b) It is just about to hit the ground back ?



Ans. (a) $\dfrac{mv_0^2 \sin 2\theta}{2}$ (b) $mv_0^2 \sin 2\theta$

Q.23

In the previous question, during the motion of particle from P to Q. Torque of gravitational force about P is :
(A) increasing (B) decreasing
(C) remains constant (D) first increasing then decreasing

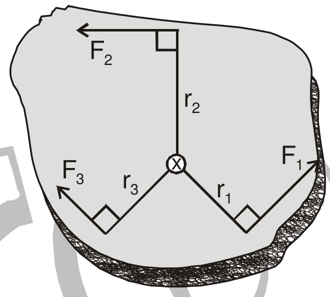
Ans. Increasing

5.2 Torque about axis :

$\vec{\tau} = \vec{r} \times \vec{F}$

where $\vec{\tau}$ = torque acting on the body about the axis of rotation.
$\vec{r}$ = position vector of the point of application of force about the axis of rotation.
$\vec{F}$ = force applied on the body.

Note : The direction of torque is calculated using right hand thumb rule and it is always perpendicular to the plane of rotation of the body.



If F_1 or F_2 is applied body applied body revolves in anti-clockwise direction
F_3 makes body revolve in clockwise direction.
If all three are applied.

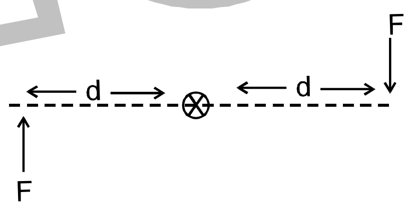
$\tau_{resultant} = F_1 r_1 + F_2 r_2 + F_3 r_3$ (in anti-clockwise direction)

Note :–

Torque produced by a force can be zero in cases. If force vector :–
(i) is parallel to the axis of rotation. (ii) passes through the axis of rotation.

5.3 Force Couple :

A pair of forces each of same magnitude and acting in opposite direction is called a force couple.
Torque due to couple = Magnitude of one force × distance between their lines of action.



Magnitude of torque = $\tau = F\,(2d)$
A couple does not exert a net force on an object even though it exerts a torque.

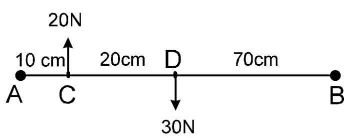
5.4 Point of Application of Force :

Point of Application of force is the point at which if nett force is assumed to be acting then it will produce same effect of both translation & rotational nature, as was produced earlier.

OR

If nett force is applied at the point of application in the opposite direction, then the body will be in equilibrium. (translational and rotational both)

Ex.21 Determine the point of application of force, when forces of 20 N & 30 N are acting on the rod as shown in figure.

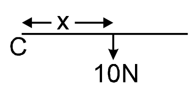


Sol. Nett force acting on the rod $F_{rel} = 10N$

Nett torque acting on the rod about point C

$\tau_c = (20 \times 0) + (30 \times 20) = 600$ clockwise

Let the point of application be at a distance x from point C

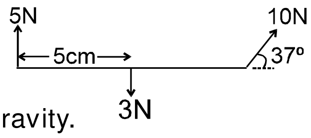


$600 = 10\,x \quad \Rightarrow \quad x = 60$ cm

$\therefore$ 70 cm from A is point of Application

Q.24

Determine the point of application of force, when forces are acting on the rod as shown in figure.



Ans. 4.375 cm left on the rod from the point where 10 N force is acting.

Note : (i) Point of application of gravitational force is known as the centre of gravity.

(ii) Centre of gravity coincides with the centre of mass if value of $\vec{g}$ is assumed to be constant.

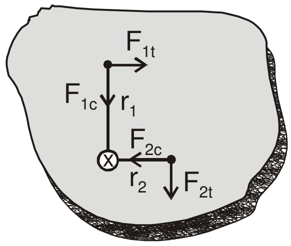
(iii) Concept of point of application of force is imaginary, as in some cases it can lie outside the body.

5.5 Relation between 'τ' & 'α ' (for hinged object or pure rotation)

$$\vec{\tau}_{ext}\big)_{Hinge} = I_{Hinge}\ \vec{\alpha}$$

Where $\vec{\tau}_{ext}\big)_{Hinge}$ = nett external torque acting on the body about Hinge point

I_{Hinge} = moment of Inertia of body about Hinge point



$F_{1t} = M_1 a_{1t} = M_1 r_1 \alpha$

$F_{2t} = M_2 a_{2t} = M_2 r_2 \alpha$

$\tau_{resultant} = F_{1t}\, r_1 + F_{2t}\, r_2 + \ldots\ldots$

$= M_1\, \alpha\, r_1^2 + M_2\, \alpha\, r_2^2 + \ldots\ldots\ldots$

$\tau_{resultant}\big)_{external} = I\,\alpha$

Rotational Kinetic Energy $= \dfrac{1}{2}.I.\omega^2$

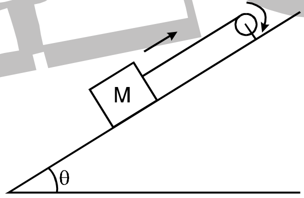
$\vec{P} = M\vec{v}_{CM}$

$\vec{F}_{external} = M\vec{a}_{CM}$

Net external force acting on the body has two parts tangential and centripetal.

$\Rightarrow \quad F_C = ma_C = m\dfrac{v^2}{r_{CM}} = m\omega^2\, r_{CM} \qquad \Rightarrow \qquad F_t = ma_t = m\alpha\, r_{CM}$

Ex.22 A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in figure. A string is wrapped round the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping ?



Sol. Suppose the deceleration of the block is a. The linear deceleration of the rim of the wheel is also a. The angular deceleration of the wheel is $\alpha = a/r$. If the tension in the string is T, the equations of motion are as follows:

$Mg \sin\theta - T = Ma \qquad$ and $\qquad Tr = I\alpha = Ia/r.$

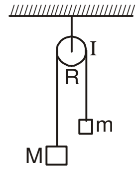
Eliminating T from these equations,

$$Mg \sin\theta - I\frac{a}{r^2} = Ma$$

giving, $\quad a = \dfrac{Mgr^2 \sin\theta}{I + Mr^2}$

The initial velocity of the block up the incline is $v = \omega\, r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2M\, r^2 \sin\theta} = \frac{(I + Mr^2)\omega^2}{2M\, g \sin\theta}$$



Ex.23 The pulley shown in figure has a moment of inertia I about its axis and its radius is R. Find the magnitude of the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.

Solution : Suppose the tension in the left string is T_1 and that in the right string in T_2. Suppose the block of mass M goes down with an acceleration α and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, $\alpha = a/R$. The equations of motion for the mass M, the mass m and the pulley are as follows :

$Mg - T_1 = Ma$ ....(i)

$T_2 - mg = ma$ ....(ii)

$T_1R - T_2R = I\alpha = I\alpha/R$ ....(iii)

Putting T_1 and T_2 from (i) and (ii) into (iii),

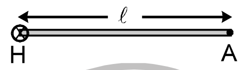
$$[(Mg - a) - m(g + a)]\, R = I\frac{a}{R}$$

which gives $a = \dfrac{(M-m)gR^3}{I+(M+m)R^2}$.

Ex.24 A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.

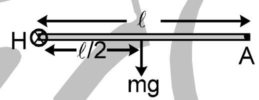
(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.



Sol. (i) $\tau_H = I_H\ \alpha$

$$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \quad \Rightarrow \quad \alpha = \frac{3g}{2\ell}$$

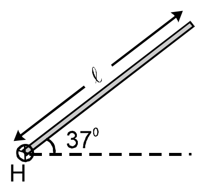


(ii) $a_{tA} = \alpha\ell = \dfrac{3g}{2\ell}.\ell = \dfrac{3g}{2}$

$a_{CA} = \omega^2 r = 0.\ell = 0$ $(\because\ \omega = 0$ just after release)

Q.25

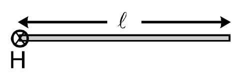
A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find angular acceleration α of the rod just after it is released from initial position making an angle of 37^0 with horizontal from rest ?



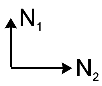
Ans. $6g / 5\ell$

Ex.25

A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find force exerted by the hinge just after the rod is released from rest, from an initial horizontal position ?



Sol. Suppose hinge exerts normal reaction in component form as shown



In vertical direction

$F_{ext} = ma_{CM}$

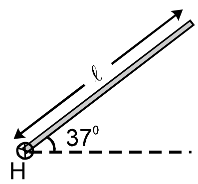
$\Rightarrow \quad mg - N_1 = m.\dfrac{3g}{4}$ (we get the value of a_{CM} from previous example)

$\Rightarrow \quad N_1 = \dfrac{mg}{4}$

In horizontal direction

$F_{ext} = ma_{CM} \quad \Rightarrow \quad N_2 = 0 \qquad (\because\ a_{CM}$ in horizontal = 0 as ω = 0 just after release).

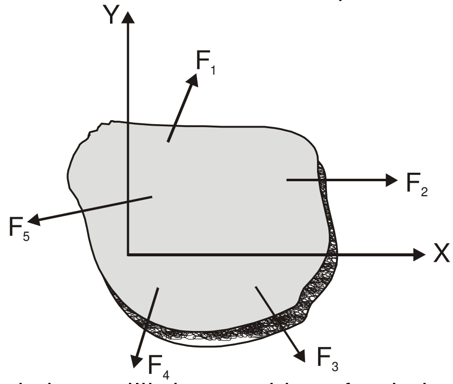
Q.26 A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find force exerted by the hinge just after the rod is released from rest, from an initial position making an angle of 37^0 with horizontal ?



Ans.

6. ROTATIONAL EQUILIBRIUM :

If nett external torque acting on the body is zero, then the body is said to be in rotational equilibrium. The centre of mass of a body remains in equilibrium if the total external force acting on the body is zero. Similarly, a body remains in rotational equilibrium if the total external torque acting on the body is zero.



For translational equilibrium.

$\Sigma F_x = 0$ .... (i)

and $\Sigma F_y = 0$ .... (ii)

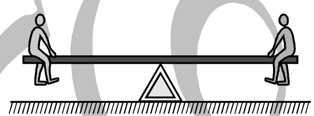
The condition of rotational equilibrium is

$\Sigma \Gamma_z = 0$

The equilibrium of a body is called stable if the body tries to regain its equilibrium position after being slightly displaced and released. It is called unstable if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in neutral equilibrium.

Ex.26

Two small kids weighing 10 kg and 15 kg are trying to balance a seesaw of total length 5.0 m, with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit ?



Solution : It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are

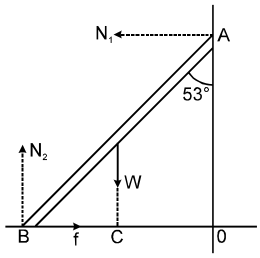
(a) (15 kg) g downward by the 15 kg kid,
(b) (10 kg) g downward by the 10 kg kid,
(c) weight of the seesaw and
(d) the normal force by the fulcrum.

Taking torques about the fulcrum,

$(15\ kg)g\ x = (10\ kg)g\ (2.5\ m)$ or $x = 1.7\ m$.

Ex.27

A uniform ladder of mass 10 kg leans against a smooth vertical wall making an angle of 53° with it. The other ends rests on a rough horizontal floor. Find the normal force and the friction force that the floor exerts on the ladder.



Sol. The forces acting on the ladder are shown in figure. They are

(a) Its weight W,
(b) normal force N_1 by the vertical wall,
(c) normal force N_2 by the floor and
(d) frictional force f by the floor.

Taking horizontal and vertical components,

$N_1 = f$ ....(i)

and $N_2 = W$ ....(ii)

Taking torque about B,

$N_1(AO) = W(CB)$

or, $N_1(AB)\cos 53° = W\,\dfrac{AB}{2}\sin 53°$ or $N_1\,\dfrac{3}{5} = \dfrac{W}{2}\,\dfrac{4}{5}$

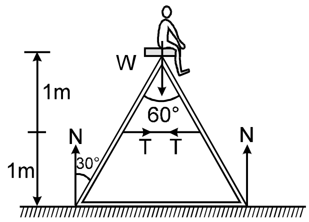
or, $N_1 = \dfrac{2}{3}W$ ....(iii)

The normal force by the floor is $N_2 = W = (10\text{ kg})\,(9.8\text{ m/s}^2) = 98\text{ N}$.

The frictional force is $f = N_1 = \dfrac{2}{3}W = 65\text{ N}$.

Ex.28

The ladder shown in figure has negligible mass and rests on a frictionless floor. The crossbar connected the two legs of the ladder at the middle. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of 80 kg. Find the constant forces exerted by the floor on each leg and the tension in the crossbar.



Solution : The forces acting on different parts are shown in figure. Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (80 kg)g and the contact force N + N = 2 N due to the floor. Thus

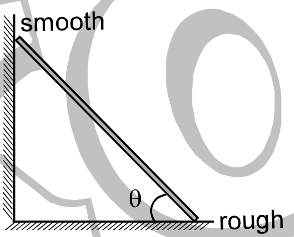
$2N = (80\text{ kg})\,g$ or $N = (40\text{ kg})\,(9.8\text{ m/s}^2) = 392\text{ N}$.

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$N\,(2\text{m})\tan 30° = T(1\text{ m})$ or $T = N\dfrac{2}{\sqrt{3}} = (392\text{ N}) \times \dfrac{2}{\sqrt{3}} = 450\text{ N}$.

Ex.29

A stationary uniform rod of mass 'm' , length 'ℓ' leans against a smooth vertical wall making an angle θ with rough horizontal floor. Find the normal force & frictional force that is exerted by the floor on the rod ?



Solution : As the rod is stationary so the linear acceleration and angular acceleration of rod is zero. i.e. $a_{cm} = 0$; $\alpha = 0$.

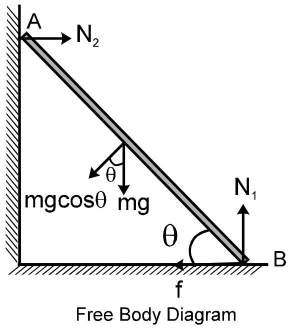
$$\left.\begin{aligned} N_2 &= f \\ N_1 &= mg \end{aligned}\right\} \because a_{cm} = 0$$

Torque about any point of the rod should also be zero. $\because \alpha = 0$

$$\tau_A = 0 \Rightarrow mg\cos\theta\,\frac{\ell}{2} + f\,\ell\sin\theta = N_1\cos\theta\,.\,\ell$$

$$N_1\cos\theta = \sin\theta\,f + \frac{mg\cos\theta}{2}$$

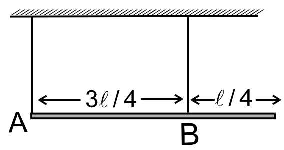
$$f = \frac{mg\cos\theta}{2\sin\theta} = \frac{mg\text{Cot}\theta}{2}$$



Free Body Diagram

Q.27

A uniform rod of length ℓ, mass m is hung from two strings of equal length from a ceiling as shown in figure . Determine the tensions in the strings ?

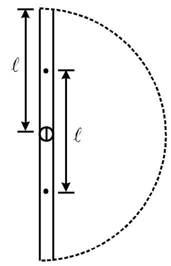


Ans. $T_A = mg/3$, $T_B = 2mg/3$

Ex.30

A uniform rod of mass m and length ℓ is kept vertical with the lower end clamped. It is slightly pushed to let it fall down under gravity. Find its angular speed when the rod is passing through its lowest position. Neglect any friction at the clamp. What will be the linear speed of the free end at this instant?

Sol. As the rod reaches its lowest position, the centre of mass is lowered by a distance ℓ. Its gravitational potential energy is decreased by mgℓ. As no energy is lost against friction, this should be equal to the increase in the kinetic energy. As the rotation occurs about the horizontal axis through the clamped end, the moment of inertia is I = m ℓ^2/3. Thus,



$$\frac{1}{2}I\omega^2 = mg\ell \qquad \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2 = mg\ell$$

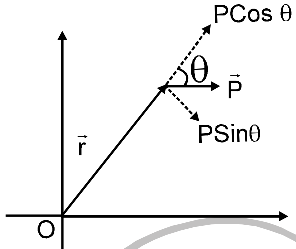
or $\omega = \sqrt{\dfrac{6g}{\ell}}$.

The linear speed of the free end is

$$v = \ell\omega = \sqrt{6g\ell}$$

7. ANGULAR MOMENTUM ($\vec{L}$)

7. 1. Angular momentum of a particle about a point.



$\vec{L} = \vec{r} \times \vec{P}$ $\Rightarrow$ L = rpsinθ

$|\vec{L}| = r_\perp \times P$

$|\vec{L}| = P_\perp \times r$

Where $\vec{P}$ = momentum of particle

$\vec{r}$ = position of vector of particle with respect to point about which angular momentum is to be calculated .

θ = angle between vectors $\vec{r}$ & $\vec{P}$

$r_\perp$ = perpendicular distance of line of motion of particle from point O.

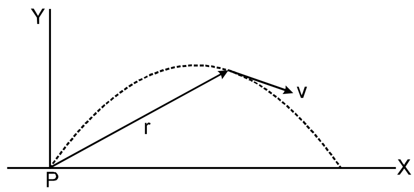
$P_\perp$ = perpendicular component of momentum.

SI unit of angular momentum is kgm²/sec.

Ex.31

A particle is projected at time t = 0 from a point P with a speed v_0 at an angle of 45° to the horizontal. Find the magnitude and the direction of the angular momentum of the particle about the point P at time t = v_0/g.

Sol. Let us take the origin at P, X-axis along the horizontal and Y-axis along the vertically upward direction as shown in figure.



For horizontal motion during the time 0 to t,

$$v_x = v_0 \cos 45° = v_0/\sqrt{2}$$

and $x = b_x t = \dfrac{v_0}{\sqrt{2}} \cdot \dfrac{v_0}{g} = \dfrac{v_0^2}{\sqrt{2}g}$.

For vertical motion,

$$v_y = v_0 \sin 45° = \frac{v_0}{\sqrt{2}} - v_0 = \frac{(1-\sqrt{2})}{\sqrt{2}}v_0 \qquad \text{and} \qquad y = (v_0 \sin 45°)\,t - \frac{1}{2}gt^2$$

$$= \frac{v_0^2}{\sqrt{2}g} - \frac{v_0^2}{2g} = \frac{v_0^2}{2g}(\sqrt{2} - 1).$$

The angular momentum of the particle at time t about the origin is

$$L = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$$

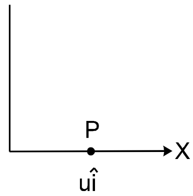
$= (\vec{i}\, x + \vec{j}\, y) \times (\vec{i}\, v_x + \vec{j}\, v_y) \qquad = \qquad m\,(\vec{k}\, x u_u - \vec{k}\, y u_x)$

$$= m\,\vec{k}\left[\left(\frac{v_0^2}{\sqrt{2}\,g}\right)\frac{v_0}{\sqrt{2}}(1-\sqrt{2}) - \frac{v_0^2}{2g}(\sqrt{2}-1)\frac{v_0}{\sqrt{2}}\right] \qquad = \qquad -\vec{k}\,\frac{mv_0^3}{2\sqrt{2}\,g}.$$

Thus, the angular momentum of the particle is $\frac{mv_0^3}{2\sqrt{2}\,g}$ in the negative Z-direction i.e., perpendicular to the plane of motion, going into the plane.

Q.28

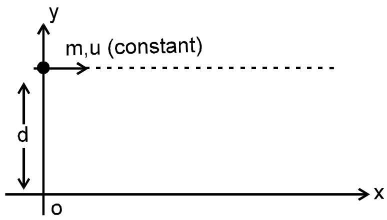
A particle of mass m starts moving from origin with a constant velocity $u\hat{i}$ find out its angular momentum about origin at this moment. What will be the answer later time ?



Ans.

Ex.32

A particle of mass 'm' starts moving from point (o,d) with a constant velocity $u\,\hat{i}$. Find out its angular momentum about origin at this moment what will be the answer at the later time?



Sol. L = mud direction is always clockwise same.

Q.29

A particle of mass 'm' is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal . Find out the angular momentum of particle about the point of projection when .

(i) it just starts its motion
(ii) it is at highest point of path.
(iii) it just strikes the ground.

Ans. (i) O ; (ii) $mu\cos\theta\,\frac{u^2\sin^2\theta}{2g}$; (iii) $mu\sin\theta\,\frac{u^2\sin 2\theta}{g}$

7.2 For system of particles :

Considering a system of particles with both external and internal forces acting we can add the angular momentum of the indivizual particles to obtain the angular momentum L.

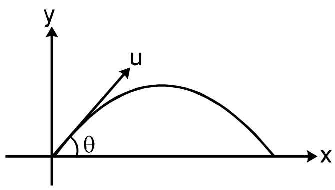
$$\vec{L} = \vec{r}_1\times\vec{p}_1 + \vec{r}_2\times\vec{p}_2 + \vec{r}_3\times\vec{p}_3 + \ldots\ldots\ldots\ldots$$

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \ldots\ldots\ldots \text{ about the same point.}$$

Ex.33

A particle of mass'm' is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal . Find out the angular momentum at any time t of particle p about :

(i) y axis
(ii) z-axis



Sol. (i) O
(ii) – 1/2 u cos θ. gt^2

7.3 Angular momentum of a rigid body rotating about fixed axis :

$\vec{L}_H = I_H\vec{\omega}$

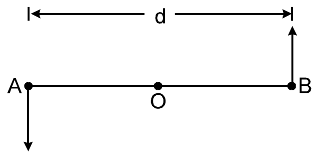
L_H = angular momentum of object about axis H.
I_H = Moment of Inertia of rigid , object about axis H.
ω = angular velocity of the object.

Ex.34

Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual balls and of the system about the axis of rotation.

Sol. Consider the situation shown in figure. The velocity of the ball A with respect to the centre O is $v = \frac{\omega d}{2}$.

The angular momentum of the ball with respect to the axis is



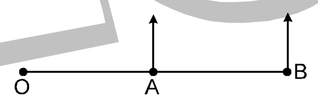
$L_1 = mvr = m\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}m\omega d^2$. The same the angular momentum

L_2 of the second ball. The angular momentum of the system is equal to sum of these two angular momenta i.e., $L = 1/2\ m\omega d^2$.

Ex.35

Two particles of mass m each are attached to a light rod of length d, one at its centre and the other at a free end. The rod is fixed at the other end and is rotated in a plane at an angular speed ω. Calculate the angular momentum of the particle at the end with respect to the particle at the centre.

Sol. The situation is shown in figure. The velocity of the particle A with respect to the fixed end O is $v_A = \omega(d/2)$ and that of B with respect to O is $v_B = \omega d$. Hence the velocity of B with respect to A is $v_B - v_A = \omega(d/2)$. The angular momentum of B with respect to A is, therefore,



$$L = mvr = m\omega\left(\frac{d}{2}\right)\frac{d}{2} = \frac{1}{4}m\omega d^2$$

along the direction perpendicular to the plane of rotation.

Ex.36

A uniform circular disc of mass 200 g and radius 4.0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the kinetic energy of the disc and its angular momentum about the axis of rotation.

Sol. The moment of inertia of the circular disc about its diameter is

$$I = \frac{1}{4}Mr^2 = \frac{1}{4}(0.200\text{ kg})(0.04\text{ m})^2 = 8.0 \times 10^{-5}\text{ kg-m}^2.$$

The kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(8.0 \times 10^{-5}\text{ kg - m}^2)(100\text{ rad}^2/\text{s}^2) = 4.0 \times 10^{-3}\text{ J}$$

and the angular momentum about the axis of rotation is

$L = I\omega = (8.0 \times 10^{-5}\text{ kg-m}^2)(10\text{ rad/s})$
$= 8.0 \times 10^{-4}\text{ kg-m}^2/\text{s} = 8.0 \times 10^{-4}\text{ J-s}.$

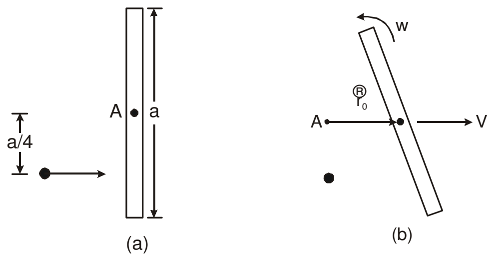
7.4 Conservation of Angular Momentum

Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

Ex.37

A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance a/4 from the centre and stops after the collision. Find (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.

Sol. The situation is shown in figure. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remains constant. Also there is no resultant external torque on the system and so the angular momentum of the system about the any line will remain constant. Suppose the velocity of the centre of the rod is V and the angular velocity about the centre is ω.



(a) The linear momentum before the collision is mv and that after the collision is MV. Thus,

$$mv = MV,$$

or $$V = \frac{m}{M}v.$$

(b) Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about AB. Initially the rod is at rest. The angular momentum of the particle about AB is

$$L = mv\,(a/4)$$

After the collision, the particle comes to rest. The angular momentum of the rod about A is

$$\vec{L} = \vec{L}_{cm} + M\vec{r}_0 \times \vec{V}$$

As $\vec{r}_0 \parallel \vec{V}$, $\quad \vec{r}_0 \times \vec{V} = 0$

Thus, $\vec{L} = \vec{L}_{cm}$

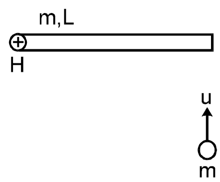
Hence the angular momentum of the rod about AB is

$$L = I\omega = \frac{M\alpha^2}{12}\omega.$$

Thus, $$\frac{mva}{4} = \frac{Ma^2}{12}\omega \text{ or, } \qquad \omega = \frac{3mv}{Ma}$$

Ex.38

A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision ?

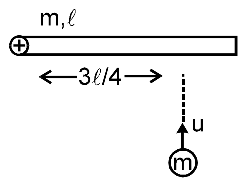


Solution : Angular momentum is conserved about H because no external force is present in horizontal plane which is producing torque about H.

$$mul = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \qquad \Rightarrow \qquad \omega = \frac{3a}{4\ell}$$

Q.30

A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at a distance of $3\ell/4$ from hinge point. Find out the angular velocity of the rod just after collision ?



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7.5 Relation between Torque and Angular Momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Torque is change in angular momentum

7.6 Impulse of Torque :

$\int \tau dt = \Delta J$ $\qquad\qquad$ $\Delta J \rightarrow$ d Charge in angular momentum.

8. Combined Translational and Rotational motion of a rigid body

If the axis of rotation is moving w.r. to ground then the motion is combined translational and rotational motion.

Kinematics :

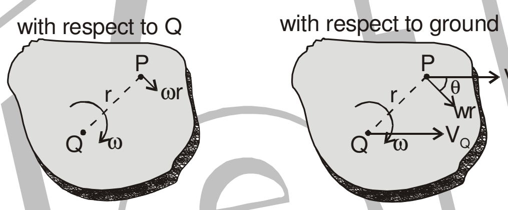
The most general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point . This is called **Chasle's Theorem**. A convenient choice of the point is the centre of mass of the body. One good example of the type of motion is rolling of a wheel.

The general motion of the body can be thought of as the result of a translation of the point Q and the motion of the body about Q. Let us choose another point P in the body with position vector $\vec{r}_P$. Let $\vec{r}_{P/Q}$ denote the position vector of P with respect to Q , then $\vec{r}_P = \vec{r}_Q + \vec{r}_{P/Q}$.

By differentiating we get , $\vec{\upsilon}_P = \vec{\upsilon}_Q + \vec{\upsilon}_{P/Q}$.

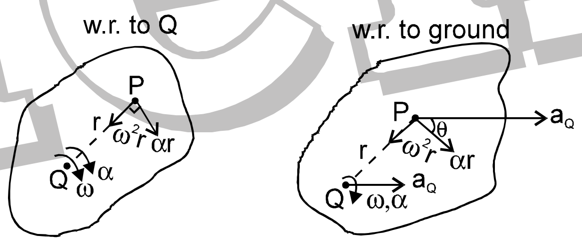
For a rigid body, the distance between the particles remain unchanged during its motion i.e. $r_{P/Q}$ = constant

For velocities



$$V_P = \sqrt{V_Q^2 + (\omega r)^2 + 2\,V_Q\,\omega r \cos\theta}$$

For acceleration :



θ, ω, α are same about every point of the body (or any other point outside which is rigidly attached to the body).

Dynamics :

$$\vec{\tau}_{cm} = I_{cm}\vec{\alpha}\ ,\ \vec{F}_{ext} = M\vec{a}_{cm} \qquad\qquad \vec{P}_{system} = M\vec{v}_{cm}\ ,$$

Total K.E. $= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\,\omega^2$

Angular momentum axis AB = $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about AB

$$\vec{L}_{AB} = I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$$

Ex.39

A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its centre moves at a speed of 2.00 cm/s. Find its kinetic energy.

Sol. As the sphere rolls without slipping on the plane surface, its angular speed about the centre is

$\omega = \frac{v_{cm}}{r}$. The kinetic energy is

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} Mv_{cm}^{\ 2} = \frac{1}{2}\cdot\frac{2}{5}Mr^2\omega^2 + \frac{1}{2}Mv_{cm}^{\ 2}$$

$$= \frac{1}{5} Mv_{cm}^{\ 2} + \frac{1}{2}Mv_{cm}^{\ 2} = \frac{7}{10}Mv_{cm}^{\ 2} = \frac{7}{10}(0.200\text{ kg})\ (0.02\text{ m/s})^2 = 5.6 \times 10^{-5}\text{ J}.$$

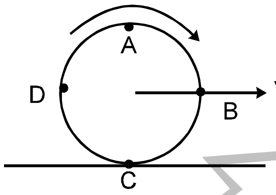
Ex.40

A wheel of perimeter 220 cm rolls on a level road at a speed of 9 km/h. How many revolutions does the wheel make per second ?

Sol. As the wheel rolls on the road, its angular speed ω about the centre and the linear speed v of the centre are related as $v = \omega r$.

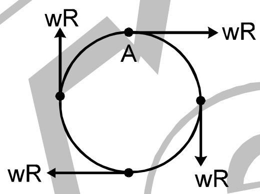
$$\therefore \qquad \omega = \frac{9\ \text{km/h}}{220\ \text{cm}/2\pi} = \frac{2\pi\times9\times10^5}{220\times3600}\ \text{rad/s.} = \frac{900}{22\times36}\ \text{rev/s} = \frac{25}{22}\ \text{rev/s.}$$

Ex.41

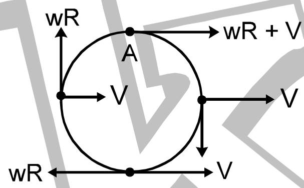


Find $\vec{V}_A$
$\vec{V}_B$
$\vec{V}_C$
$\vec{V}_D$

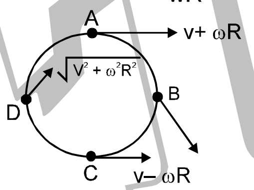
Solution : wrt. centre



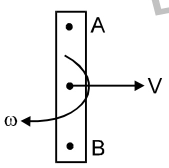
wrt. ground



Result



Q.31



$V_A = ?$, $V_B = ?$

Ans.

9. PURE ROLLING :

Ex.42

A cylinder is released from rest from the top of an incline of inclination θ and length ℓ. If the cylinder rolls without slipping, what will be its speed when it reaches the bottom ?

Sol. Let the mass of the cylinder be m and its radius r. Suppose the linear speed of the cylinder when it reaches the bottom is v. As the cylinder rolls without slipping, its angular speed about its axis is $\omega = v/r$. The kinetic energy at the bottom will be

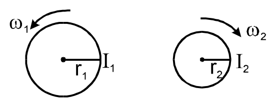
$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 \qquad = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = \frac{1}{4}\ mv^2 + \frac{1}{2}\ mv^2 = \frac{3}{4}\ mv^2.$$

This should be equal to the loss of potential energy $mg\ell\sin\theta$. Thus,

$$\frac{3}{4}mv^2 = mg\ell \sin\theta \qquad \text{or} \qquad v = \sqrt{\frac{4}{3}g\ell\sin\theta}.$$

Ex.43

Figure shows two cylinders of radii r_1 and r_2 having moments of inertia I_1 and I_2 about their respective axes. Initially, the cylinders rotate about their axes with angular speed ω_1 and ω_2 as shown in the figure. The cylinders are moved closed to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Find the angular speeds of the cylinders after the slipping ceases.



Sol. When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If ω'_1 and ω'_2 be the respective angular speeds, we have

$\omega'_1 r_1$ and $\omega'_2 r_2$ ....(i)

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t, the torque on the first cylinder is fr_1 and that on the second is fr_2. Assuming $\omega_1 > \omega_2$, the corresponding angular impulses are $-fr_1t$ and fr_2t, We, there fore, have

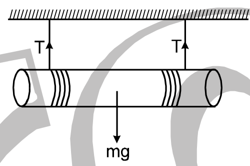
$$-f r_1 t = I_1(\omega'_1 - \omega_1) \qquad \text{and} \qquad fr_2 t = I_2(\omega'_2 - \omega_2)$$

or, $$-\frac{I_1}{r_1}(\omega'_1 - \omega_1) = \frac{I_2}{r_2}(\omega'_2 - \omega_2) \qquad ....(ii)$$

Solving (i) and (ii) $$\omega'_1 = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_2 \qquad \text{and} \qquad \omega'_2 = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_1.$$

Ex.44

A cylinder of mass m is suspended through two strings wrapped around it as shown in figure. Find (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance h.



Sol. The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$mg - 2T - ma$ ....(i)

and for the motion about the centre of mass, it is

$$2\,Tr\left(\frac{1}{2}mr^2\alpha\right) = \frac{1}{2}\,mra \qquad \text{or} \qquad 2T = \frac{1}{2} = ma.$$

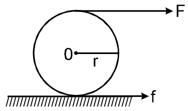
From (i) and (ii), $$a = \frac{2}{3}g \text{ and } T = \frac{mg}{6}.$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance h is given by

$$v^2 = 2\left(\frac{2}{3}g\right)h \qquad \text{or} \qquad v = \sqrt{\frac{4\,gh}{3}}.$$

Ex.45

A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre of the sphere.



Sol. The situation is shown in figure. As the force F rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere will act towards right. Let r be the radius of the sphere and a be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre,

$F + f = ma$ ....(i)

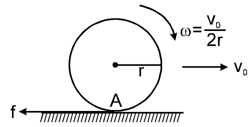
and for the rotational motion about the centre,

$$Fr - f\,r = I\,\alpha = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right) \qquad \text{or,} \qquad F - f = \frac{2}{5}ma, \qquad ....(iii)$$

From (i) and (ii), $\quad 2F = \frac{7}{5}ma \quad$ or $\quad a = \frac{10F}{7m}$.

Ex.46

A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and rotational velocity about the centre $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.



Sol. Velocity of the centre = v_0 and the angular velocity about the centre = $\frac{v_0}{2r}$. Thus $v_0 > \omega_0 r$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic, its value is $\mu N = \mu Mg$ and the sphere will be decelerated by $a_{cm} = f/M$. Hence,

$$v(t) = v_0 - \frac{f}{M}t. \qquad ....(i)$$

This friction will also have a torque Γ = fr about the centre. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the centre will be

$$\alpha = f\,\frac{r}{(2/5)Mr^2} = \frac{5f}{2Mr}$$

and the clockwise angular velocity at time t will be $\quad \omega(t) = \omega_0 + \frac{5f}{2Mr}t = \frac{v_0}{2r} + \frac{5f}{2Mr}t.$

Pure rolling starts when $v(t) = r\,\omega(t)$ $\quad$ i.e., $\quad v(t) = \frac{v_0}{2} + \frac{5f}{2M}t. \qquad ....(ii)$

Eliminating t from (i) and (ii), $\quad \frac{5}{2}v(t) + v(t) = \frac{5}{2}v_0 + \frac{v_0}{2} \quad$ or, $\quad v(t) = \frac{2}{7} \times 3v_0 = \frac{6}{7}v_0.$

Thus, the sphere rolls with translational velocity $6v_0/7$ in the forward direction.

